ON THE INJECTION OF AN ELECTRICALLY-CONDUCTING LIQUID OR GAS INTO A BOUNDARY LAYER IN THE PRESENCE OF A MAGNETIC FIELD

(O VDUVANII V POGRANICHNYI SLOI V PRISUTSTVII MAGNITNOGO POLYA ELEKTROPROVODNOI ZHIDKOSTI ILI GAZA)

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The problem is considered of the injection of an electrically-conducting fluid into the boundary layer formed on the surface of a flat plate, in the presence of a magnetic field perpendicular to the surface of the plate.

1. Suppose a plane semi-infinite plate immersed in a gas flow with constant temperature T_∞, density ρ_∞ and velocity u_∞. Through the surface of the plate an electrically-conducting liquid (gas) is introduced, forming a thin layer on the plate as it is entrained by the external flow. The flow proceeds in a magnetic field H, perpendicular to the plane of the plate (Fig. 1). We choose a system of coordinates oriented with axis x^0 along, and axis y^0 perpendicular to, the surface of the plate. The layer on the wall and the exterior flow are separated by a surface of discontinuity, on which the physical-chemical properties of the material change. The layer on the surface and all the quantities in it we shall designate by the index 2, and the part of the boundary layer relating to the exterior flow we shall designate by the index 1.

We go over to dimensionless variables according to the formulas

$$x^\circ = lx, \quad u^\circ = u_\infty u, \quad \rho^\circ = \rho_\infty \rho, \quad T^\circ = T_\infty T, \quad H^\circ = H_* H(x)$$

$$y^\circ = \frac{l}{\sqrt{R}}\, y, \quad v^\circ = \frac{u_\infty}{\sqrt{R}}\, v, \quad \eta^\circ = \eta_\infty \eta, \quad \sigma^\circ = \sigma_* \sigma, \quad k^\circ = k_\infty k \qquad (1.1)$$

Here u and v are the velocities in the x- and y-directions respectively, η is the coefficient of dynamic viscosity, k is the coefficient of thermal conductivity, σ is the electrical conductivity of the medium, l is the characteristic dimension along the plate, H_* is the characteristic intensity of the magnetic field, H° is the normal component of the magnetic field on the wall, $R = \rho_\infty u_\infty l/\eta_\infty$ is the Reynolds number. The dimensional quantities η_∞, k_∞ refer to the inflowing stream, σ_* is the characteristic conductivity of the injected fluid and quantities with the superscript $^\circ$ have dimensions.

We assume that the conductivity of the medium in region 1 may be neglected in comparison with the conductivity of the injected fluid, and the magnetic Reynolds number, based on the length of the boundary layer, is of order unity, whereas the quantity $1/\sqrt{R}$ is small:

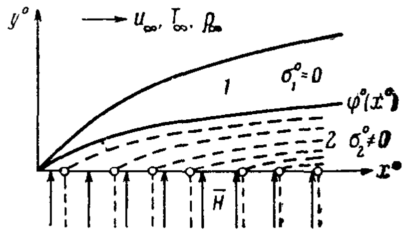

Fig. 1.

$$R_m = \frac{u_\infty l\sigma_*}{c^2} 4\pi = O(1), \qquad \frac{1}{\sqrt{R}} = o(1)$$

In dimensionless variables, the equations of motion in region 1 may be written

$$\rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} = \frac{\partial}{\partial y} \eta \frac{\partial u}{\partial y}, \qquad \frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} = 0$$
$$\rho u \frac{\partial T}{\partial x} + \rho v \frac{\partial T}{\partial y} = m_1^2 \eta \left(\frac{\partial u}{\partial y}\right)^2 + \frac{1}{P_1} \frac{\partial}{\partial y} \eta \frac{\partial T}{\partial y} \tag{1.2}$$

In region 2 the motion is described by the system

$$\rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} = \frac{\partial}{\partial y} \eta \frac{\partial u}{\partial y} - \gamma\sigma H^2(x) u, \qquad \frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} = 0$$
$$\rho u \frac{\partial T}{\partial x} + \rho v \frac{\partial T}{\partial y} = m_2^2 \eta \left(\frac{\partial u}{\partial y}\right)^2 + \frac{1}{P_2} \frac{\partial}{\partial y} \eta \frac{\partial T}{\partial y} + m_2^2 \gamma\sigma H^2(x) u^2 \tag{1.3}$$

In the systems (1.2) and (1.3) the following constant parameters are introduced:

$$\gamma = \frac{\sigma_* H_*^2 l}{c^2 \rho_\infty u_\infty}, \qquad P_i = \frac{c_{p_i} \eta_i^\circ}{k_i^\circ}, \qquad m_i^2 = \frac{u_\infty^2}{c_{p_i} T_\infty} \qquad (i = 1, 2)$$

The equations of motion are supplemented by conditions on the surface of the plate, the line of discontinuity $\Phi(x)$ and the surface of the boundary layer. On the outer surface, we have $u_1 = 1$, $T_1 = 1$. On the surface of the plate we assume that $u_2 = 0$ and that the temperature conditions and the time rate of introduction of the fluid are known. Finally, it is possible to show that when the magnetic permeabilities of the two media are the same ($\mu_1 = \mu_2 = 1$) the relations on the surface of discontinuity can be reduced to the form used in [1], in which the solution of the problem of fluid introduction without the magnetic field is found:

$$u_1\tan\beta - v_1 = u_2\tan\beta - v_2 = 0$$

$$\left(\eta\frac{\partial u}{\partial y}\right)_1 = \left(\eta\frac{\partial u}{\partial y}\right)_2, \quad \left(k\frac{\partial T}{\partial y}\right)_1 = \left(k\frac{\partial T}{\partial y}\right)_2, \quad u_1 = u_2, \quad T_1 = T_2, \quad \tan\beta = \frac{d\varphi}{dx} \qquad (1.4)$$

The problem posed is self-similar if the magnetic field and the time rate of introduction of the fluid are proportional to $1/\sqrt{x}$

$$H^\circ = H_*\frac{1}{\sqrt{x}}, \qquad \rho(x, 0)\, v(x, 0) = \frac{\rho_\infty u_\infty}{\sqrt{R_x}}\frac{C}{2}, \qquad R_x = \frac{\rho_\infty u_\infty x^\circ}{\eta_\infty}$$

We introduce the Blasius variable $\zeta = 1/\sqrt{x}$ and the function $\omega(u) = \eta\, du/d\zeta$; Equations (1.2) and (1.3) can be reduced to the system of ordinary differential equations

$$\omega'' + K_1^2\frac{u}{2\omega} = 0, \qquad T'' + T'\frac{\omega'}{\omega}(1 - P_1) + (Pm^2)_1 = 0 \qquad \text{for } u^* \leqslant u \leqslant 1 \qquad (1.5)$$

$$\omega'' + K_2^2\frac{u}{2\omega} - \gamma\frac{d}{du}\frac{\psi u}{\omega} = 0$$

$$T'' + T'\left[\frac{\omega'}{\omega}(1 - P_2) + \frac{P_2\gamma\psi u}{\omega^2}\right] + (Pm^2)_2\left(1 + \frac{\gamma\psi u^2}{\omega^2}\right) = 0 \qquad \text{for } 0 \leqslant u \leqslant u^*$$

Here u^* is the velocity along the line of discontinuity

$$T = T(u), \qquad T' = \frac{dT}{du}; \qquad K_1^2 = K_1^2(T) = \frac{\rho_1^\circ\eta_1^\circ}{\rho_\infty\eta_\infty}$$

$$\omega' = \frac{d\omega}{du}; \qquad \psi = \psi(T) = \frac{\sigma_2^\circ\eta_2^\circ}{\sigma_*\eta_\infty}, \qquad K_2^2 = K_2^2(T) = \frac{\rho_2^\circ\eta_2^\circ}{\rho_\infty\eta_\infty}$$

The functions K_1^2, K_2^2 and ψ depend only on the temperature because the pressure is constant in the entire boundary layer. The system (1.5) is supplemented by the relations

$$\begin{array}{llll} T_1 = 1, & \omega_1 = 0 & & \text{for } u = 1 \\ \omega_1' = 0, & \omega_2' = \dfrac{\gamma\psi u}{\omega_2}, & \omega_1 = \omega_2 & \\ NT_1' = T_2', & T_1 = T_2 & \left(N = \dfrac{k_1^\circ\eta_2^\circ}{\eta_1^\circ k_2^\circ}\right) & \text{for } u = u^* \\ \omega_2' = {}^1/_2\, C, & T = T_w & & \text{for } u = 0 \end{array} \qquad (1.6)$$

The nine conditions (1.6) are sufficient for the solution of the two systems (1.5), each of which is of the fourth order, and for the determination of the velocity on the surface of discontinuity.

2. The solution of the system (1.5) with boundary conditions (1.6) was obtained in the case of constant values of K_1^2, K_2^2 and ψ where the parameter K_2^2 has a large magnitude. For this the coefficient of dynamic

viscosity in region 1 should be proportional to the temperature ($K_1^2 = 1$), and the motion in region 2 should take place with negligible discontinuity in temperature (then the density, conductivity and coefficient of dynamic viscosity may be considered constant). The condition $K_2^2 = K^2 \gg 1$ is satisfied if one require that the density and coefficient of dynamic viscosity in region 2 are larger than in region 1. Then

$$\gamma\psi = \frac{\sigma_* H_*^2 l \eta_2^\circ}{c^2 \rho_\infty u_\infty \eta_\infty} = K\gamma_*, \qquad \frac{1}{K} = o(1), \qquad \gamma_* = O(1)$$

For these assumed conditions, it is possible to solve the dynamic problem (determination of the velocity field) separately from the thermal problem (determination of the temperature field). From the system (1.5) we obtain the following closed system of equations:

$$\omega'' + \frac{u}{2\omega} = 0 \quad \text{for } u^* \leqslant u \leqslant 1; \qquad \omega'' + \frac{K^2 u}{2\omega} - \gamma_* K \frac{d}{du}\frac{u}{\omega} = 0 \quad \text{for } 0 \leqslant u \leqslant u^* \tag{2.1}$$

$$\omega_1(1) = 0, \quad \omega_2'(0) = \frac{1}{2}C; \quad \omega_1' = 0, \quad \omega_2' = \gamma_* K \frac{u}{\omega_2}, \quad \omega_1 = \omega_2 \quad \text{for } u = u^*$$

We introduce expressions for the coefficient of friction and the overall drag of the plate. For the force acting on unit area of the plate we have the expression

$$F = \left(\eta^\circ \frac{\partial u^\circ}{\partial y^\circ}\right)_w + \int_0^{\varphi^\circ(x^\circ)} \frac{\sigma^\circ H^{\circ 2} u^\circ}{c^2}\, dy^\circ$$

Introducing the friction coefficient and the total drag by the formulas

$$c_f = \frac{(\eta^\circ \partial u^\circ / \partial y^\circ)_w}{{}^1/_2\, \rho_\infty u_\infty^2}, \qquad c_d = \frac{F}{{}^1/_2\, \rho_\infty u_\infty^2}$$

we obtain

$$\frac{c_f \sqrt{R_x}}{2} = \omega_2(0), \qquad \frac{c_d \sqrt{R_x}}{2} = \omega_2(0) + \frac{C\gamma_*}{K}$$

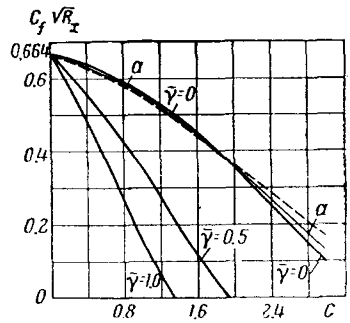

Fig. 2.

We make use of the fact that the parameter K is large. The solution of the system (2.1) can be obtained by expansion in a power series in K^{-1} as described in [2]. Omitting the unwieldy details, we present the final results. For the friction coefficient and the total drag we obtain

$$c_f \sqrt{R_x} = 0.664 - 0.5431\frac{C^{3/2}}{K} - \frac{2\gamma_* C}{K} + \frac{0.0526\, C^3 - 0.1974\, C}{K^2} + 0.2863\,\gamma_* \frac{C^{5/2}}{K^2} + O(K^{-3}) \tag{2.2}$$

$$c_d \sqrt{R_x} = 0.664 - 0.5431\frac{C^{3/2}}{K} + \frac{0.0526\, C^3 - 0.1974\, C}{K^2} + 0.2863\,\gamma_* \frac{C^{3/2}}{K^2} + O(K^{-3}) \tag{2.3}$$

For the parameter of magneto-gasdynamical interaction γ^* equal to zero, we obtain the solution of the problem of injection of fluid without the magnetic field. This problem was solved in [1] by use of a numerical integration of the equations. In the absence of a magnetic field and fluid injection Formula (2.2) gives the classical solution of the boundary-layer equation. In Fig. 2 is shown the dependence of $c_f\sqrt{R_x}$ on the injection constant C for $K = 5$ and different values of $\gamma_* = \overline{\gamma}$, computed by taking account of first-order terms in K^{-1}. The curve "a" is also presented, as computed for $\overline{\gamma} = 0$, taking account of second-order terms in K^{-1}. The dotted curve is the curve obtained in [1]. In spite of the relatively small size of K, for values of the constant C in the range from 0 to 2, excellent agreement with the exact solution is obtained when only the first approximation is used. This is evidence of the effectiveness of the proposed method of expansion in terms of the small parameter $1/K$. As is evident from Fig. 2, the presence of the magnetic field leads to a decrease in surface friction. The first-order total drag, as follows from Formula (2.3), coincides with the friction drag in the absence of the magnetic field. The increase in drag by reason of the cutting of magnetic force lines by the stream is compensated by the decrease in the surface friction. Further, calculation of the terms of order K^{-2} indicates that the overall drag of the plate with fluid injection in the presence of a magnetic field increases. However, it does not exceed the drag of the plate in question without injection and without taking account of the magnetic field.

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